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Minimizing the loss of entanglement under dimensional reduction

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Abstract. We investigate the possibility of transforming, under local operations and classical communication, a general bipartite quantum state on a $d_A \times d_B$ tensor-product space into a final state in 2×2 dimensions, while maintaining as much entanglement as possible. For pure states, we prove that Nielsen's theorem provides the optimal protocol, and we present quantitative results on the degree of entanglement before and after the dimensional reduction. For mixed states, we identify a protocol that we argue is optimal for isotropic and Werner states. In the literature, it has been conjectured that some Werner states are bound entangled and in support of this conjecture our protocol gives final states without entanglement for this class of states. For all other entangled Werner states and for all entangled isotropic states some degree of free entanglement is maintained. In this sense, our protocol may be used to discriminate between bound and free entanglement.

PACS. 03.67.Mn Entanglement production, characterization, and manipulation – 42.50.Dv Nonclassical states of the electromagnetic field, including entangled photon states; quantum state engineering and measurements – 03.65.Ud Entanglement and quantum nonlocality (e.g. EPR paradox, Bell's inequalities, GHZ states, etc.)

1 Introduction

Entanglement plays a crucial role in quantum teleportation, quantum computing, and quantum cryptography. Many of the protocols used for these purposes use maximally entangled pairs of qubits, i.e., 2×2 systems (see, for example, the monograph by Nielsen and Chuang [1]). Here we consider the situation where a bipartite state is prepared in 3×3 or generally $d_A \times d_B$ dimensions and we transform the state, using local quantum operations and classical communication (LQCC), into a 2×2 state under the requirement that the final state possesses as much entanglement as possible.

For pure states, we will show that the optimal dimensional reduction protocol can be derived from a recipe for pure state transformations described by Nielsen [2], and we present an explicit implementation for our problem. For mixed states, we consider a protocol for the highly symmetric Werner [3] and isotropic [4] states. While the general problem of transforming mixed states remains unsolved, we argue that our protocol is optimal for the states considered.

As an example of a less symmetric mixed state, we consider the $\rho(\alpha)$ introduced by the Horodecki's (see, e.g., Ref. [5]). Also for these states the protocol maintains entanglement under dimensional reduction and it correctly identifies the border between free and bound entangled states.

The paper is organized as follows. In Section 2, we recall the definitions of separable and entangled states, and we introduce the entanglement measure to be used: entanglement of formation (EOF). In Section 3, we describe the reduction protocol for pure states and prove that it is optimal in the sense that it maximizes the EOF in the final 2×2 system. In Section 3, we also present our quantitative results on the EOF before and after the dimensional reduction of pure states. In Section 4, we describe the reduction protocol for mixed states and we present calculations on isotropic, Werner, and $\rho(\alpha)$ states.

2 Entanglement and an entanglement measure

A quantum state is either separable or entangled. A pure state $\rho = |\Psi\rangle\langle\Psi|$ on a tensor product space, $|\Psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$, is separable if it can be written as a product of states belonging to each Hilbert space individually, i.e.,

$$|\Psi\rangle = |\Psi_A\rangle \otimes |\Psi_B\rangle. \quad (1)$$

To see if a given pure state $|\Psi\rangle$ is separable it is sufficient to consider its Schmidt decomposition (see, e.g., Ref. [1]).

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If only a single Schmidt coefficient is non-zero, the state is separable. Otherwise it is entangled.

A mixed state ρ is separable if it can be written as a convex sum of direct products of states belonging to $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively

$$\rho = \sum_j w_j \rho_j^A \otimes \rho_j^B, \quad \sum_j w_j = 1, \quad w_j \geq 0, \qquad (2)$$

where ρ_j^A and ρ_j^B are hermitian and positive semidefinite. A state which is not separable is entangled.

It is generally difficult to decide whether a given mixed state is entangled or not. A necessary condition for a state ρ to be separable is that it has a positive partial transpose $\rho^{T_A} \geq 0$ (all eigenvalues are non-negative) with respect to a given product basis ($\langle ij|\rho^{T_A}|kl\rangle = \langle kj|\rho|il\rangle$) [6]. Partially transposing the separable density matrix in equation (2) means transposing all ρ_j^A leading to another density matrix which is positive by construction.

Entangled states are classified in two categories: free entangled states and bound entangled states. This classification is related to the question whether it is possible to obtain a single or more maximally entangled 2×2 states from several copies of ρ by means of LQCC, i.e., whether a given quantum state ρ is distillable or not. Free entangled states are distillable (and have a negative partial transpose). Bound entangled states cannot be distilled. In references [7,8] it was conjectured that there exist bound entangled states whose partial transpose have a negative eigenvalue. No proof exists of this conjecture. If it holds true, however, it will have far-reaching consequences as discussed in reference [9].

To quantify the degree of entanglement of a given state ρ, various measures of entanglement have been developed. In this work, we use entanglement of formation (EOF) as entanglement measure [10]. EOF vanishes for separable states and is positive for entangled states. This of course means that if one could calculate EOF for a given state, the question of separability or entanglement would be solved. Unfortunately it turns out to be very difficult to evaluate the EOF except in special cases.

For pure states EOF may be evaluated by the von Neuman entropy [10]

$$E(\Psi) = -\mathrm{tr}(\rho_A \ln \rho_A) = -\mathrm{tr}(\rho_B \ln \rho_B) \qquad (3)$$

where $\rho_A = \mathrm{tr}_B(|\Psi\rangle\langle\Psi|)$ and $\rho_B = \mathrm{tr}_A(|\Psi\rangle\langle\Psi|)$ are the reduced density matrices on $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively, and ln is the natural logarithm. EOF is $\ln d$ for a maximally entangled state in $d \times d$ dimensions. If $\log_2$ is used instead of ln, EOF of equation (3) asymptotically equals the number of maximally entangled qubits necessary to create the state by means of LQCC [10].

There exists an analytical formula for the EOF of 2×2 mixed states [11,12]. First one writes ρ as a 4×4 matrix in the product state representation $\{|11\rangle, |12\rangle, |21\rangle, |22\rangle\}$, and one computes $\tilde{\rho} = O\rho^* O^T$ where O is a 4×4 matrix with 4 non-zero entries $O_{14} = O_{41} = 1$ and $O_{23} = O_{32} = -1$. Let λ_j be the eigenvalues of $\sqrt{\sqrt{\rho}\tilde{\rho}\sqrt{\rho}}$ in decreasing order. Then the concurrence is defined as

$$C(\rho) = \max(0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4), \qquad (4)$$

and in terms of $C(\rho)$, the EOF is given by

$$E(\rho) = H_2\left(\frac{1}{2}\left[1 + \sqrt{1 - C^2(\rho)}\right]\right), \qquad (5)$$

where H_2 is

$$H_2(x) = -x \ln x - (1-x)\ln(1-x). \qquad (6)$$

For general mixed states EOF is the minimum over all decompositions into pure states [10]

$$E(\rho) = \min \sum_{j=1}^{k} p_j E(\Psi_j), \quad \rho = \sum_{j=1}^{k} p_j |\Psi_j\rangle\langle\Psi_j|. \qquad (7)$$

The EOF of a mixed state is thus uniquely defined by the cheapest way to produce it (in terms of the number of maximally entangled qubits). In the general case one has to evaluate EOF by equation (7) directly. This implies taking the minimum over infinitely many decompositions of ρ. Clearly this exposes the difficulty of evaluating EOF in general. Despite the difficulties, there exist recipes for the evaluation of EOF, and we have successfully implemented the algorithm of reference [13].

The purpose of this paper is to study transformations of states from $d_A \times d_B$ to 2×2, and in particular to study the conservation or loss of entanglement under such a dimensional reduction as evaluated by our numerical application of equation (7) and the analytical expression of equation (5).

3 Reduction protocol for pure states

Initially, we start with a bipartite state $|\psi\rangle$ on a $d_A \times d_B$ dimensional tensor product Hilbert space. Using LQCC, we want to transform $|\psi\rangle$ into a bipartite final state in a lower dimension (2×2) which possesses as much entanglement as possible. The final state could be either a state $|\phi\rangle$ which we obtain with certainty or an ensemble of states $\{p_j, |\phi_j\rangle\}$ where state $|\phi_j\rangle$ is obtained with probability p_j.

The transformation properties of pure states have been studied previously in the literature. In reference [2], Nielsen gives the necessary and sufficient condition for when it is possible to transform $|\psi\rangle$ into $|\phi\rangle$ with certainty under LQCC. In reference [14], Vidal gives the condition for transforming $|\psi\rangle$ into a state $|\phi\rangle$ with a probability p, and in reference [15], Jonathan and Plenio give the condition for transforming a state into an ensemble $\{p_j, |\phi_j\rangle\}$.

3.1 Majorization, reduction to pure and to mixed states

In the Schmidt decomposition of pure states the dimension of the system is $d = \min(d_A, d_B)$. We now let $\mathbf{x} = (x_1, ..., x_d)$ and $\mathbf{y} = (y_1, ..., y_d)$ be the eigenvalues in decreasing order of $\rho_\psi = \mathrm{tr}_B(|\psi\rangle\langle\psi|)$ and $\rho_\phi = \mathrm{tr}_B(|\phi\rangle\langle\phi|)$.

The protocols to transform pure states all use the concept of majorization. The set of eigenvalues $\mathbf{x}$ is said to be majorized by $\mathbf{y}$ ($\mathbf{x} \prec \mathbf{y}$) if the following equations are fulfilled

$$\sum_{j=1}^{k} x_j \leq \sum_{j=1}^{k} y_j, \quad k = 1, ..., d$$
$$\sum_{j=1}^{d} x_j = \sum_{j=1}^{d} y_j. \tag{8}$$

We want to identify the highest possible EOF for the final 2×2 state. Nielsen's theorem [2] states that one can convert $|\psi\rangle$ into $|\phi\rangle$ *if and only if* $\mathbf{x} \prec \mathbf{y}$. We consider the case where $|\phi\rangle$ is in a two-dimensional space. This means $y_3 = ... = y_d = 0$ and the majorization criterion reduces to a single inequality

$$x_1 \leq y_1. \tag{9}$$

If $x_1 \leq 1/2$, we can transform $|\psi\rangle$ to the maximally entangled 2×2 state with $y_1 = y_2 = 1/2$. If $x_1 > 1/2$, the final state with the highest EOF has $y_1 = x_1$. We now address the question whether it is better to end up with an ensemble of entangled states $\{p_j, |\phi_j\rangle\}$. The work by Jonathan and Plenio [15] shows that one can convert $|\psi\rangle$ into $\{p_j, |\phi_j\rangle\}$ *if and only if* $\mathbf{x} \prec \sum_j p_j \mathbf{y}_j$. (Note that the transformation of Vidal [14] is contained as a special case in the work of Jonathan and Plenio [15].) For reduction to 2×2 states the majorization criterion reads $x_1 \leq \sum p_j y_{1_j}$, and since the eigenvalues are ordered $y_{1j} \geq y_{2j}$, we also have $\sum p_j y_{1j} \geq 1/2$. Since the most entangled 2×2 pure state $|\phi\rangle$ obtained according to Nielsen's theorem has $y_1 = \max(x_1, 1/2)$, also $y_1 \leq \sum p_j y_{1_j}$, and therefore $|\phi\rangle$ can be transformed into $\{p_j, |\phi_j\rangle\}$. Because the EOF cannot increase under this transformation the highest EOF of a pure state obtained by Nielsen's theorem is larger than or equal to the EOF of any ensemble $\{p_j, |\phi_j\rangle\}$.

3.2 State conversion, a simple example

The virtue of Nielsen's work was to prove the relationship between majorization and state conversion. The proof is constructive in the sense that when the majorization condition is fulfilled, it explicitly presents the measurements and unitary operations to be applied on the separate parts of the quantum system to convert a state $|\psi\rangle$ into state $|\phi\rangle$.

It is worthwhile to show with a simple example how the maximally entangled state in 3×3, $|\Psi^+\rangle = (|11\rangle + |22\rangle + |33\rangle)/\sqrt{3}$, can be transformed to a maximally entangled 2×2 state $(|11\rangle + |22\rangle)/\sqrt{2}$. We first observe that the eigenvalues of the initial and final states are $x_1 = x_2 = x_3 = 1/3$ and $y_1 = y_2 = 1/2$, respectively, so the transformation is possible. If we measure on $\mathcal{H}_A$, whether the system is in state $|3\rangle$ or not, with a probability of 1/3 the system is projected into the product state $|33\rangle$, and with probability 2/3, the system is projected into the desired maximally entangled state $(|11\rangle + |22\rangle)/\sqrt{2}$. To obtain the maximally entangled state with certainty, we have to perform a generalized measurement on $\mathcal{H}_A$ which projects the system into one of the three non-orthogonal two-dimensional subspaces spanned by $\{|1\rangle, |2\rangle\}$, by $\{|1\rangle, |3\rangle\}$, or by $\{|2\rangle, |3\rangle\}$. Such a measurement can be formulated as a positive operator-valued measure (POVM), and it can be implemented by coupling the system represented by $\mathcal{H}_A$ to an auxiliary quantum system and by performing a normal von Neumann measurement on this other system. As a result of the local quantum measurement, the state $|\Psi^+\rangle$ is transformed into one of three maximally entangled 2×2 states $(|ii\rangle + |jj\rangle)/\sqrt{2}$, $(i, j) = (1, 2), (1, 3), (2, 3)$, and local unitary operations can subsequently (if necessary) transform this state into $(|11\rangle + |22\rangle)/\sqrt{2}$.

In the general case it is a more tedious task to identify the appropriate general measurement and the necessary unitary operations; see reference [2] for a complete description.

In the following subsection, we present the results of transformations of general pure states, and in Section 4 we turn to mixed states, which we shall transform with the protocol outlined above for the maximally entangled state.

3.3 Transforming from 3×3 to 2×2

We will start by considering the transformation of 3×3 states into 2×2 states. In the initial state, we then have three eigenvalues $\mathbf{x} = (x_1, x_2, 1 - x_1 - x_2)$ and in the final state we have $\mathbf{y} = (y_1, 1 - y_1, 0)$, and the majorization criterion is given in equation (9). The initial and final EOF are calculated from equation (3) with the results

$$\begin{aligned} E(\psi) &= - x_1 \ln x_1 - x_2 \ln x_2 \\ &\quad - (1 - x_1 - x_2) \ln(1 - x_1 - x_2) \\ E(\phi) &= - y_1 \ln y_1 - (1 - y_1) \ln(1 - y_1). \end{aligned} \tag{10}$$

The eigenvalues x_1 and x_2 are varied between the possible values $x_1 \in [1/3, 1]$ and $x_2 \in [(1 - x_1)/2, \min(1 - x_1, x_1)]$, and we maximize the final EOF by choosing $y_1 = \max(x_1, 1/2)$. The initial and final EOF are calculated and plotted in Figure 1. The states on the curve with slope 1 are initially in 2 dimensions with $x_3 = 0$. The states on the upper horizontal line can be transformed to the maximally entangled state because $x_1 \leq 1/2$. On the first piece of the horizontal line where $E(\psi) \in [\ln 2, \frac{3}{2} \ln 2]$, $x_1 = 1/2$; on the rest of the line to the right of the arrow in the figure $x_1 \leq 1/2$. On the lower curve of the hatched area $x_2 = x_3$. The final EOF is at most $E(\phi) = \ln 2$, but as the figure clearly shows, there are many states which cannot be transformed to the maximally entangled 2×2 state even though they have initial EOF larger than $\ln 2$.

3.4 Transforming from d $\times$ d to 2×2

When we consider the dimensional reduction from $d_A \times d_B$ to 2×2 states we have $d = \min(d_A, d_B)$ in the Schmidt basis and we write the eigenvalues in decreasing order as $x = (x_1, ..., x_{d-1}, 1 - x_1 - ... - x_{d-1})$ and $y = (y_1, 1 - y_1, 0, ..., 0)$. The majorization criterion is again given in equation (9),

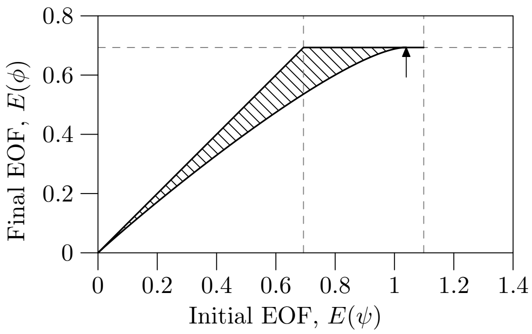

Fig. 1. EOF before and after transformation of a 3×3 state, $|\psi\rangle$, into the most entangled 2×2 state, $|\phi\rangle$. The two vertical dashed lines correspond to initial EOF of $E(\psi) = \ln 2$ and $E(\psi) = \ln 3$, respectively. The horizontal line corresponds to final EOF of $E(\phi) = \ln 2$. The arrow indicates where the lower curve reaches the horizontal line which corresponds to an initial EOF of $E(\psi) = (3/2)\ln 2$.

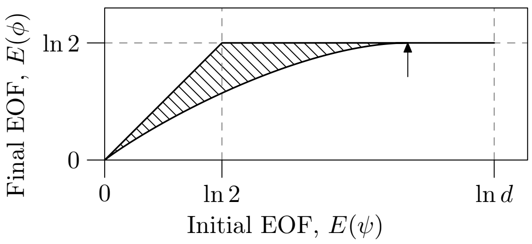

Fig. 2. EOF before and after transformation of a $d \times d$ state into the most entangled 2×2 state. The arrow indicates where the lower curve reaches the horizontal line which corresponds to an initial EOF of $E(\psi) = \ln 2 + \ln(d-1)/2$.

and the initial and final EOF can be calculated by a straightforward generalization of equation (10). To maximize the final EOF we must choose $y_1 = \max(x_1, 1/2)$. Varying x_1, ... x_{d-1} over the possible values leads to the results shown in Figure 2.

4 Reduction protocol for mixed states

Mixed states are described by density matrices instead of wave functions. Despite large efforts, a generalization of Nielsen's theorem to mixed states has not been found, and one of the motivations for the present work was, indeed, to look at the dimensional reduction as a less general problem for which a solution may be found. For isotropic and for Werner states we believe that we have found a protocol that minimizes the loss of entanglement as we transform from $d_A \times d_B$ to 2×2 states. As we shall see, the protocol also provides interesting results for other states.

The isotropic state is a convex mixture of the maximally entangled state $|\Psi^+\rangle = (1/\sqrt{d})\sum_{j=1}^{d}|jj\rangle$ and the identity matrix $\mathbb{I}$ and it reads [4]

$$\rho_F = \frac{1-F}{d^2-1}\left(\mathbb{I} - |\Psi^+\rangle\langle\Psi^+|\right) + F|\Psi^+\rangle\langle\Psi^+|,\ 0 \le F \le 1. \tag{11}$$

For $F = 1$, ρ_F is the maximally entangled state $|\Psi^+\rangle$.

Werner states [3] are linear combinations of the identity and the flip operator $\mathbb{F} = \sum_{i,j=1}^{d}|ij\rangle\langle ji|$

$$\begin{aligned}\rho_p &= p\rho_- + (1-p)\rho_+,\ 0 \le p \le 1\\ \rho_\pm &= \frac{1}{d(d\pm 1)}(\mathbb{I} \pm \mathbb{F}).\end{aligned} \tag{12}$$

For $p = 1$, ρ_p is the maximally entangled Werner state.

4.1 Transformation of isotropic and Werner states

A transformation protocol is specified by a set of operations to be carried out on the physical system in an experiment. It therefore makes perfect sense to apply a pure state protocol, even when the initial state is a mixed state, by simply carrying out the measurements and unitary transformations pertaining to a given initial $|\psi\rangle$ and final $|\phi\rangle$ on the system.

When we transform isotropic and Werner states, we use the pure state protocol for $|\Psi^+\rangle$ described in Section 3.2. $|\Psi^+\rangle$ is the entangled component of the isotropic state and is therefore a natural choice for the pure state determining the transformation. The maximal entangled Werner state can be written as a sum of spin singlets

$$\rho_{p=1} = \frac{1}{d(d-1)}\sum_{i<j}(|ij\rangle - |ji\rangle)(\langle ij| - \langle ji|). \tag{13}$$

We would expect that the optimal way to transform these states into qubits is to use a POVM $\{M_{ij} = (1/\sqrt{d-1})(|i\rangle\langle i| + |j\rangle\langle j|)\}$ built out of $d(d-1)/2$ projections onto subspaces of $\mathcal{H}_A$ spanned by any two vectors $|i\rangle$ and $|j\rangle$. This is precisely the projections used in the optimum transformation of $|\Psi^+\rangle$ cf. Section 3.2.

For mixed states the first measurement of the protocol transforms from the state $d_A \times d_B$ to $2 \times d_B$ and a measurement on $\mathcal{H}_B$ is necessary to ensure that the system resides in the desired two-dimensional subspace.

The isotropic state and the Werner state are very symmetric, and the effect of the projection into two dimensional subspaces of $\mathcal{H}_A$ and $\mathcal{H}_B$ can be analyzed by considering the case of projection on the spaces spanned by $\{|1\rangle_A, |2\rangle_A\}$ and $\{|1\rangle_B, |2\rangle_B\}$. In case of the isotropic state, the $d\times d$ identity matrix is projected into the 2×2 identity matrix, and $|\Psi^+\rangle$ is projected into $\sqrt{2/d}(|11\rangle + |22\rangle)/\sqrt{2}$, and hence the isotropic state is converted into a 2×2 isotropic state. Writing this state on the form of equation (11), with $d = 2$, we obtain the relationship between the parameter F_d of the $d \times d$ state and F_2 of the reduced state

$$F_2 = \frac{F_d(2d^2-d) + d - 2}{F_d(2d^2-4d) + 4d - 2}, \tag{14}$$

$$F_2 = \frac{15F_3 + 1}{6F_3 + 10}, \quad \text{for } d = 3. \tag{15}$$

In case of the Werner state, we also find simple transformations of the $\mathbb{I}$ and the $\mathbb{F}$ components into 2×2, and we identify the resulting state as a 2×2 Werner state with the parameter relationship

$$p_2 = \frac{p_d(d+1)}{3(d-1)-(2d-4)p_d}, \tag{16}$$

$$p_2 = \frac{2p_3}{3-p_3}, \quad \text{for } d=3. \tag{17}$$

4.2 Loss of entanglement

Let us now consider the loss of entanglement caused by the dimensional reduction.

4.2.1 Isotropic states

A $d \times d$ isotropic state is separable for $0 \le F < 1/d$. For $1/d \le F \le 1$, the EOF is given by the convex hull of the function $H_2(\gamma) + [1-\gamma]\ln(d-1)$ [16], where

$$\gamma = \frac{1}{d}\left[\sqrt{F} + \sqrt{(d-1)(1-F)}\right]^2. \tag{18}$$

In reference [16] it is shown for $d=3$ and conjectured in the general case that the EOF is given by

$$E(\rho_F) = \begin{cases} 0, & 0 \le F \le \frac{1}{d}, \\ H_2(\gamma) + [1-\gamma]\ln(d-1), & \frac{1}{d} \le F \le \frac{4(d-1)}{d^2}, \\ \frac{d\ln(d-1)}{d-2}(F-1) + \ln d, & \frac{4(d-1)}{d^2} \le F \le 1. \end{cases} \tag{19}$$

Following equation (14) we observe that the final state will be entangled ($F_2 > 1/2$) as long as $F_d > 1/d$. Since there is no bound entanglement in 2×2 we conclude that there are no bound entangled isotropic states. When we compute the EOF of the final state we have observed, both for the isotropic and for the Werner states, described below, that the EOF of the $2 \times d$ state determined numerically after the measurement on $\mathcal{H}_A$ equals the EOF of the final 2×2 state with parameters in equations (14, 16) multiplied with the probability that the measurement on $\mathcal{H}_B$ projects the system into the appropriate subspace. Figure 3 shows the initial vs. the final EOF for the transformation of isotropic states. The highest EOF of $\ln d$ is reduced to $\ln 2$, and as the curves for $d = 2, 3$, and 4 show, there is an almost linear scaling of the EOF by the factor $\ln 2/\ln d$.

4.2.2 Werner states

Werner states (Eq. (12)) have the entanglement of formation [17],

$$E(\rho_p) = H_2\left[\frac{1}{2}\left(1 - \sqrt{1-(1-2p)^2}\right)\right] \tag{20}$$

valid for $1/2 < p \le 1$. If $0 \le p \le 1/2$, the state is separable.

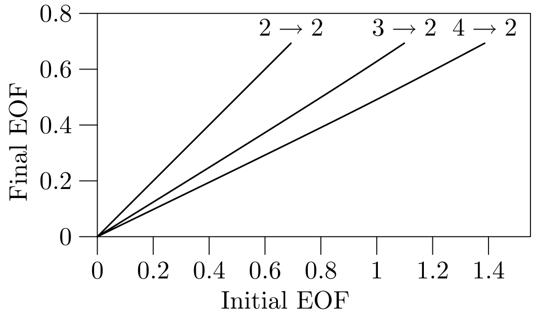

Fig. 3. EOF before and after transformation of isotropic states for different dimensions of the initial state.

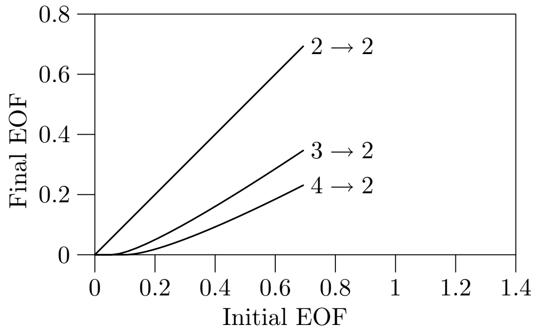

Fig. 4. EOF before and after transformation of Werner states for different dimensions of the initial state.

Following equation (16) we identify an interesting range of parameters

$$\frac{1}{2} < p_d \le \frac{3(d-1)}{2(2d-1)} \quad \left(\frac{1}{2} < p \le \frac{3}{5} \text{ for } d = 3\right)$$

for which the state is entangled, but the final state after reduction to 2×2 is separable ($p_2 \le 1/2$). This supports the conjecture in references [7,8] that these states are bound entangled, because bound entangled states cannot be transformed into free entangled states and there exist no bound entangled 2×2 states.

For the maximally entangled Werner state, after the measurement on $\mathcal{H}_A$, the possible outcomes on $\mathcal{H}_B$ will either lead to a spin singlet $(|ij\rangle - |ji\rangle)(\langle ij| - \langle ji|)$ with probability $1/(d-1)$ or a separable vector in the space spanned by $|ik\rangle$ or $|jk\rangle$, $k \ne i, j$. The EOF of the final state after conversion is thus $\ln 2/(d-1)$, which is smaller than the initial $E(\rho) = \ln 2$ for any value of d. In Figure 4, we plot the initial vs. the final EOF for the transformation of Werner states. The endpoint of the curves are at $(\ln 2, \ln 2/(d-1))$ as announced, and for $d > 2$ we observe the existence of entangled states in $d \times d$ which loose all entanglement and transform to separable states in 2×2 — candidates for bound entangled states.

Figure 5 summarizes the results for transformation from 3×3 to 2×2, and it provides a comparison with the pure state case, showing that our mixed state examples loose relatively more EOF than the pure states.

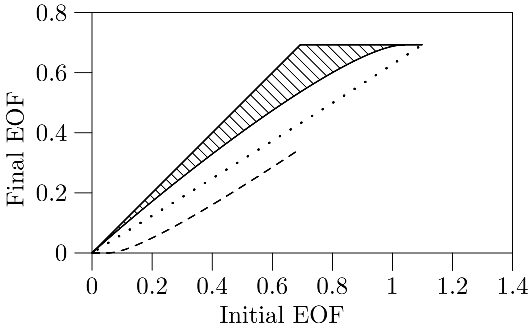

Fig. 5. EOF before and after transformation of pure (hatched region), isotropic (dotted line), and Werner (dashed line) 3×3 states into 2×2 states.

4.3 $\rho(\alpha)$-states

As an example of a less symmetric state we have studied the 3×3-dimensional density matrix $\rho(\alpha) \in \mathcal{H}_A \times \mathcal{H}_B$ described in reference [5]

$$\rho(\alpha) = \frac{2}{7}|\Psi^+\rangle\langle\Psi^+| + \frac{\alpha}{7}\sigma_+ + \frac{5-\alpha}{7}\mathbb{F}\sigma_+\mathbb{F}, \tag{21}$$

where $0 \leq \alpha \leq 5$, $\sigma_+ = (|12\rangle\langle 12| + |23\rangle\langle 23| + |31\rangle\langle 31|)/3$, and $\mathbb{F}$ is the flip operator (see Eq. (12)).

To transform the state $\rho(\alpha)$ from 3×3 to 2×2 dimensions, we have used the measurements based on $|\Psi^+\rangle$ as we did in the transformation of isotropic and Werner states. The entanglement properties of $\rho(\alpha)$ are symmetric around $\alpha = 2.5$ and for $0 \leq \alpha < 1$ the state is free entangled, for $1 \leq \alpha < 2$ it is bound entangled, and for $2 \leq \alpha \leq 3$ it is separable. Following the protocol, we have derived an analytical formula for the concurrence of the final state

$$C(\rho) = \max\left(0, \frac{4}{9} - \frac{2\sqrt{\alpha(5-\alpha)}}{9}\right). \tag{22}$$

Since the EOF of equation (5) is greater than zero if and only if the concurrence is greater than zero, it follows from equation (22) that bound entangled states indeed loose all entanglement and that free entangled states have positive final entanglement. This shows that for $\rho(\alpha)$ our protocol distinguishes between free and bound entanglement. Figure 6 displays the results. Indeed we see that in the regions with initial free entangled states, the final states are also entangled. Contrary, in the regions with initial bound entangled or separable states, the final states are separable. The values for the initial EOF are calculated using the numerical algorithm presented in reference [13], which according to equation (7) provides an upper bound for the EOF. We observe that in the regions $1.8 \leq \alpha \leq 2$ and $3 \leq \alpha \leq 3.2$, our numerical results provide values of EOF $< 10^{-4}$. It would be difficult to ascertain the entanglement of these states without the proof of the Horodecki's [5] that $\rho(\alpha)$ is entangled for these values.

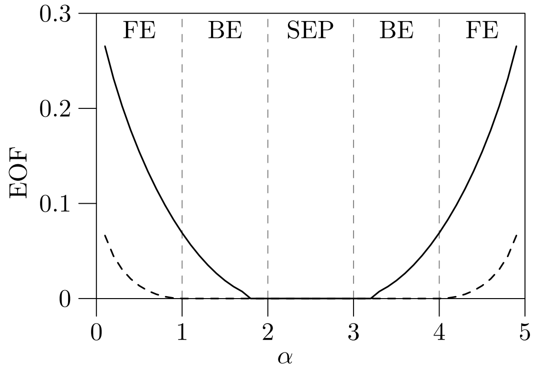

Fig. 6. Initial (full line) and final (dashed line) EOF for the mixed state $\rho(\alpha)$ as a function of the parameter α under the dimensional reduction $3\times 3 \rightarrow 2\times 2$. The vertical lines separate the parameter regions where the initial state is free entangled (FE), bound entangled (BE) and separable (SEP).

5 Conclusions and outlook

We have considered the dimensional reduction of quantum states under local operations and classical communication with the aim of obtaining a final state which possesses as much entanglement as possible. For pure states we have shown that the optimal protocol can be derived from the pure state transformation theorem by Nielsen [2]. For mixed states, we have considered isotropic and Werner states where we have argued that our protocol is optimal. Our protocol supports the conjecture in references [7,8] that some of the Werner states with negative partial transpose are bound entangled. We have also used our protocol on another state for which it is probably not optimal. Nevertheless it distinguishes between free and bound entangled states in the sense that all free entangled states are still entangled when we have transformed them into 2×2 dimensions.

The results presented in this paper bring about a series of questions for further studies. Clearly it is important to have optimal protocols for maximization of the EOF of a quantum state. The problem of transforming a given state ρ into a 2×2 state with the constraint that the final state be as entangled as possible can be formulated in terms of a highly nonlinear optimization problem over generalized measurement operators. This problem is very hard to solve as in principle an infinite number of generalized measurements need to be considered. Interestingly, already transformation protocols derived from pure states, as the one applied in this paper, have room for improvement. For this class of protocols, once the initial pure state is specified, the POVM's are explicitly determined by Nielsen's theorem. In the present work, we used the protocol derived for a specific maximally entangled pure state to transform mixed states. This strategy cannot be optimal in general, and an interesting task for a given ρ, could be to identify the state vector $|\Psi\rangle$ and the associated POVM's and unitary operations that transform ρ optimally, i.e., the pure state protocol which maximizes the EOF of the final 2×2 state. To this end, we note that equation (7) provides a decomposition of ρ into pure states $|\Psi_i\rangle$. We have studied this decomposition and we have found that the pure states typically have more or less equal weights and similar EOF's, i.e., no preferred pure state is singled out. A more fruitful approach worthy of further analytical and numerical studies thus seems to apply the measurements and operations on ρ with the protocols obtained for a suitably parametrized set of pure states, and through a variational procedure, seek to optimize the EOF of the final state.

Because bound entangled states necessarily loose their EOF when dimensionally reduced to 2×2 states (where all entangled states are free entangled), such an optimal protocol would allow one to decide more firmly whether a given initial state is bound (one-copy undistillable) or free entangled.

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